Chow Rings

Introductory Presentation of Intersection Theory (towards the Hirzebruch/Grothendieck–Riemann–Roch theorem)

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April 27, 2026

Abstract

In this presentation, we introduce the notion of Chow groups for separated scheme of finite type over an algebraically closed field of characteristic zero. We briefly mention the ring structure and its geometric interpretation using intersections for smooth quasiprojective varieties, and then briefly introduce the notion of proper pushforward and flat pullback. Finally, we construct the intersection between a divisor and an element from the Chow group.

Standing Assumptions: Throughout the following, by *scheme* we mean a separated scheme of finite type over an algebraically closed field k of characteristic zero, and morphisms between schemes are assumed to be k-morphisms. Under this assumption, we see that such scheme is automatically Noetherian[1]. Under this assumption, we see that a morphism between schemes is automatically separated (Apply [Har77, II.4.6.(e)]). We also see that a morphism between schemes is automatically quasicompact (since the domain scheme is Noetherian, and hence its topological space is Noetherian, and open subsets of Noetherian spaces are quasicompact [Vak25, 3.6.U]). We also see that a morphism between schemes is automatically of finite type (Apply [Har77, II.Ex3.13.(f)]).

A variety is an integral (i.e. reduced and irreducible) scheme.

1 The Basic Idea

The roots of intersection theory come from enumerative geometry, i.e., combinatorial questions of "counting the number of intersections". As the subject evolved, people became interested in the remarkable fact that such intersection numbers become invariant under perturbations (if one defines them "correctly"). It turns out that a lot of the work is on "how to define the intersection numbers correctly", and then the subject found its nature in the framework of algebraic geometry: instead of working with lines in the affine plane, we move to (quasi-)projective spaces and require smoothness. This gives a rich framework to handle (geometric) intersection counting problems using tools from algebraic geometry.

In this talk, we will introduce the Chow group (Chow ring in the case of smooth quasi-projective varieties), mention the geometric intersection interpretation, and compute a few examples.

2 Introducing the Chow Group

Let X be a scheme. Let $Z(X)$ be the free abelian group generated by all the closed subvarieties of X (i.e. a closed irreducible subset of X, equipped with the induced reduced subscheme structure [Vak25, 9.4.9] or [Har77, Example II.3.2.6]). One can grade $Z(X)$ by the dimension of the subvariety, say, let $Z_k(X)$ be the free abelian group generated by all the k-dimensional closed subvarieties of X.

As an extension, to each closed subscheme Y of X, we remove the nonreduced structure and consider Y_{red} as a reduced subscheme, then decompose it into a union of irreducible components, say, $Y_1, Y_2, \ldots, Y_n$. Let $\eta_1, \ldots, \eta_n$ be their generic points with the nonreduced structures. We denote by $\langle Y\rangle \in Z(X)$ the element $\sum_{i=1}^{n} \ell_{\mathscr{O}_{Y,\eta_i}}(\mathscr{O}_{Y,\eta_i})\langle Y_i\rangle$. It is the cycle associated to Y.

The *Chow group* is the group of *cycles modulo rational equivalence*. The idea is similar to one of homology: *cycles modulo boundaries*. In this analogue, we define rational equivalence as the following:

Definition (Rational Equivalence)**.** *Let X be a scheme. Let $Y, Y' \subseteq X$ be closed subschemes of X. We say that Y is (pre)rationally equivalent to Y' if there exists a closed subvariety $\Phi \subseteq \mathbb{P}^1 \times X$ such that*

1. *The subvariety Φ is not entirely contained in a single fiber $\{t\} \times X$ for any $t \in \mathbb{P}^1$.*
2. *There exists $t_0, t_1 \in \mathbb{P}^1$ such that the fiber $\langle \Phi \cap (\{t_0\} \times X)\rangle = \langle \{t_0\} \times Y\rangle$, and the fiber $\langle \Phi \cap (\{t_1\} \times X)\rangle = \langle \{t_1\} \times Y'\rangle$.*

The subgroup $\mathrm{Rat}(X) \subseteq Z(X)$ is defined as the subgroup generated by $\langle Y\rangle - \langle Y'\rangle$ for $Y, Y' \subseteq X$ closed subschemes whenever Y is (pre)rationally equivalent to Y'. We say that closed subschemes $Y, Y' \subseteq X$ are rationally equivalent if $\langle Y\rangle - \langle Y'\rangle \in \mathrm{Rat}(X)$.

Now we're ready to define the Chow group.

Definition (Chow Group)**.** *Let X be a scheme. The Chow group of X, denoted by $A(X)$, is defined as $Z(X)/\operatorname{Rat}(X)$.*

Remark (Notation)**.** *For a closed subvariety $Y \subseteq X$, we write $\langle Y\rangle \in Z(X)$ as its cycle, and we write $[Y] \in A(X)$ as its equivalence class in the Chow group. For any cycle $\alpha \in Z(X)$, we write $[\alpha]$ for its class. Note that $[Y] = [\langle Y\rangle]$.*

[1] Because by definition of finite type over k, the scheme X can be covered by finitely many open affines $U_j \subseteq X$ with $U_j = \operatorname{Spec}(A_j)$ where each A_j is a finitely generated k-algebra (hence Noetherian once seen as a quotient of $k[x_1, x_2, \ldots, x_n]$ by some ideal; the fact that $k[x_1, \ldots, x_n]$ is Noetherian comes from Hilbert's basis theorem).

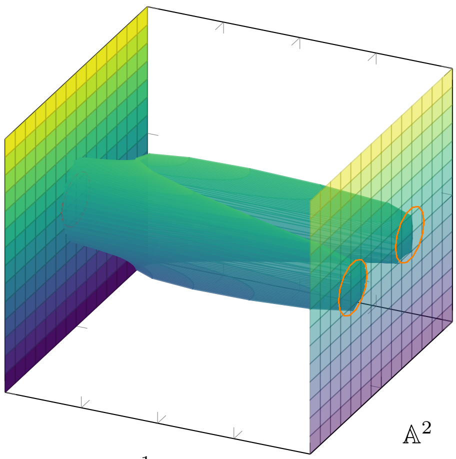
Proposition 2.1. *We can grade the Chow group $A(X)$ into $\bigoplus_{k\geq 0} A_k(X)$ by the dimension (i.e. $A_k(X) := Z_k(X)/(\mathrm{Rat}(X) \cap Z_k(X))$).*

Proof. (Taken from [EH16, Proposition 1.4.]) It is enough to show that for closed subvarieties $Y, Y' \subseteq X$, if they are rationally equivalent, then they have the same dimension.

Indeed, suppose Y and Y' are rationally equivalent, say, let $\Phi \subseteq \mathbb{P}^1 \times X$ be a closed subvariety not contained in a single fiber and let $t_0, t_1 \in \mathbb{P}^1$ such that $\{t_0\} \times Y = \Phi \cap (\{t_0\} \times X)$ and $\{t_1\} \times Y = \Phi \cap (\{t_1\} \times X)$.

In an appropriate open set $t_0 \in \mathbb{A}^1 \subseteq \mathbb{P}^1$, we see that $\Phi \cap (\{t_0\} \times X)$ is simply the zero locus of $t - t_0$ in $\Phi \cap (\mathbb{A}^1 \times X)$, hence has codimension 1 in Φ. Similarly $\Phi \cap (\{t_1\} \times X)$ has codimension 1 in Φ. Since Φ is irreducible, the dimension is pure, and so both $\{t_0\} \times Y$ and $\{t_1\} \times Y$ have the same dimension. □

Example 2.2 (Example of Rational Equivalence)**.** *Consider the following example. Fix $k = \mathbb{C}$ and let $X = \mathbb{A}^2$. Let $Y_1 := V_{\mathbb{A}^2}(x^2 + y^2 - 1)$ be the unit circle. Let $Y_2 := V_{\mathbb{A}^2}(((x-2)^2 + y^2 - 1)((x+2)^2 + y^2 - 1))$ be a union of two circles. One defines $\Phi \subseteq \mathbb{P}^1 \times X$ as $\Phi := V_{\mathbb{P}^1 \times \mathbb{A}^2}(s(x^2 + y^2 - 1) + t((x-2)^2 + y^2 - 1)((x+2)^2 + y^2 - 1))$ where $(s : t) \in \mathbb{P}^1$ is the variable in the projective line, to see that the fiber at $(1 : 0)$ is just Y_1 and the fiber at $(0 : 1)$ is just Y_2.*

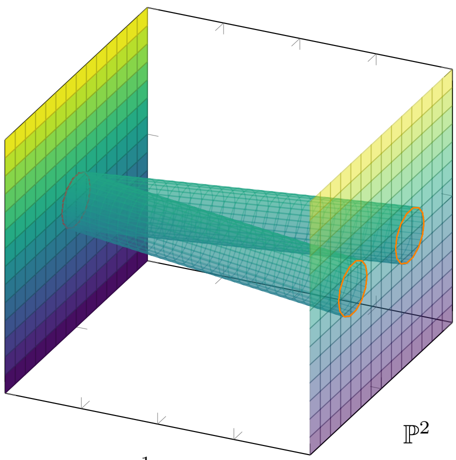


(One still has to check that Φ is irreducible)

However, one might try to do the same thing for the circles in $\mathbb{P}^2$. Consider the following.

Example 2.3 (Nonexample of Rational Equivalence)**.** *Fix $k = \mathbb{C}$ and let $X := \mathbb{P}^2$. Let $Y_1 := V_{\mathbb{P}^2}(x_1^2 + x_2^2 - x_0^2)$, and let $Y_2 := V_{\mathbb{P}^2}(((x_1 - 2x_0)^2 + x_2^2 - x_0^2)((x_1 + 2x_0)^2 + x_2^2 - x_0^2))$. One checks that these two are not rationally equivalent!*

Side note: I initially thought that the following figure (with $\Phi := V_{\mathbb{P}^1 \times \mathbb{P}^2}(((sx_1 - 2tx_0)^2 + s^2x_2^2 - s^2x_0^2)((sx_1 + 2tx_0)^2 + s^2x_2^2 - s^2x_0^2))$ would work, but then I see now that it doesn't work because Φ is not irreducible, and even if something like this works, one has to be careful and count the scheme-theoretic multiplicity on the fibers.



Another equivalent formulation of rational equivalence is to define something similar to *boundaries* in the following. Recall the settings of order of vanishing of a rational function at a subvariety as follows.

Definition. *Let V be a variety. Let $W \subseteq V$ be a closed subvariety of codimension one. Let η be the generic point for W. Let $R := \mathscr{O}_{V,\eta}$ be the stalk. It is Noetherian, and so if $f \in R$ then the module $R/(f)$ has finite length. It is a domain, since V is irreducible. It has dimension one, since W has codimension one. This means it is a 1-dimensional Noetherian local domain. Let $\operatorname{ord}_W(f) := \ell_R(R/(f))$, for $f \in R$. Then we can extend this to $\operatorname{ord}_W \colon \operatorname{Frac}(\mathscr{O}_{V,\eta})^\times \to \mathbb{Z}$ which is a group homomorphism, by sending $\frac{f}{g}$ to $\operatorname{ord}_W(f) - \operatorname{ord}_W(g)$. This is well-defined by additivity of length and the following short exact sequence:*

$$0 \to R/(f) \to R/(fg) \to R/(g) \to 0.$$

Identifying $\operatorname{Frac}(\mathscr{O}_{V,\eta})^\times$ with $K(V)^\times$, we get $\operatorname{ord}_W \colon K(V)^\times \to \mathbb{Z}$, the order of vanishing of a rational function at a closed subvariety.

Now, for a scheme X, for a closed subvariety V of dimension $k+1$, and for a nonzero rational function $\varphi \in K(V)^\times$, we can define

$$\operatorname{div}(\varphi) := \sum_{\substack{Y \subseteq V \\ \text{closed subvariety of} \\ \text{codimension one}}} \operatorname{ord}_Y(\varphi)\langle Y \rangle \in Z_k(X).$$

The fact that the sum is finite can be shown using a minor adaptation of [Har77, II.6.1].

Let $B_k(X)$ be the subgroup of $Z_k(X)$ generated by the image of div. Then, we have the following alternative characterization. This is the definition of the Chow groups given in [Gat18].

Proposition 2.4. *Let X be a scheme. For all $k \geq 0$, $\operatorname{Rat}(X) \cap Z_k(X) = B_k(X)$. In particular, $A_k(X) = Z_k(X)/B_k(X)$.*

Proof. See [Ful98, Proposition 1.6.]. □

Now we're specifically interested in the case of a smooth quasi-projective variety. This is where the Chow group turns into a ring once an appropriate product structure is equipped.

3 Intersection and the Product Structure

We introduce some basic notions about transverse intersection and construct the product structure.

Definition (Transverse Intersection)**.** *Let X be a variety. Let A and B be closed subvarieties of X. Let $p \in X$ be a closed point. We say that A intersects B transversely at p if*

1. *The varieties A, B, and X are all smooth at p,*
2. *The tangent space T_pA and the tangent space T_pB together span T_pX, i.e. $T_pA + T_pB = T_pX$.*

We say that a property $\mathcal{P}$ holds generally in X if there exists an open dense $U \subseteq X$ such that $\forall p \in U, \mathcal{P}(p)$ holds.

Definition (Generically Transverse Intersection)**.** *Let X be a variety. Let A and B be closed subvarieties of X. We say that A intersects B generically transversely (and say A and B are generically transverse) if for each component C of $A \cap B$, they intersect generally transversely (i.e. for each component C of $A \cap B$ there exists an open dense $U \subseteq C$ such that for all closed point $p \in U$, A intersects B transversely at p).*

Theorem 3.1. *Let X be a smooth quasi-projective variety. There is a unique product structure $\cdot \colon A(X) \times A(X) \to A(X)$ satisfying the commutative ring axioms and the following property:*

If two subvarieties A and B of X are generically transverse, then $[A] \cdot [B] = [A \cap B]$.

Once this theorem is true, it turns the Chow group $A(X)$ into a commutative ring, and we call this the Chow ring. This theorem is a bit hard to prove. One way is to first establish the following *moving lemma* first, but the modern method is more general, and is detailed in the Fulton–MacPherson approach in [Ful98].

Lemma 3.2 (The Moving Lemma [EH16, Theorem 1.6.]). *Let X be a smooth quasi-projective variety. Then*

(a) For every $\alpha, \beta \in A(X)$, there are generically transverse cycles $A, B \in Z(X)$ with $[A] = \alpha$ and $[B] = \beta$.

(b) The class $[A \cap B]$ is independent of the choice of such cycles A and B.

Before moving on, let us compute some explicit examples which turns out to be useful. Observe:

Lemma 3.3. *Let X be a variety of dimension n. Then $B_n(X) = 0$. In particular, the* fundamental class $[X] \in A(X)$ *is never zero.* □

Proposition 3.4. $A(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n]$

Sketch (following [EH16, Proposition 1.13.]). First, let us show that any closed subvariety of $\mathbb{A}^n$ (except itself) is rationally equivalent to $\varnothing$. Indeed, let $Y \subsetneq \mathbb{A}^n$ be a closed subvariety. Without loss of generality, assume Y doesn't contain the origin (otherwise just translate a point of $\mathbb{A}^n \setminus Y$ to the origin; such action is a rational equivalence). Let

$$\Phi := V_{\mathbb{P}^1 \times \mathbb{A}^n}\left(\left\{ f\left(\frac{z_1 s}{t}, \frac{z_2 s}{t}, \ldots, \frac{z_n s}{t}\right) t^{\deg f} : f \in I_{\mathbb{A}^n}(Y) \right\}\right),$$

then the fiber at $(s : t) = (0 : 1)$ is $\{(0 : 1)\} \times V_{\mathbb{A}^n}(\{f(0, 0, 0, \ldots, 0) : f \in I_{\mathbb{A}^n}(Y)\})$, but since the origin is not in Y, we see that there is some f with $f(0, 0, \ldots, 0) \neq 0$ so such set is empty, hence $\Phi_{(0:1)} = \varnothing$. Now, the fiber at $(s : t) = (1 : 1)$ is just $\{(1 : 1)\} \times V_{\mathbb{A}^n}(\{f : f \in I_{\mathbb{A}^n}(Y)\}) = \{(1 : 1)\} \times V(I(Y)) = \{(1 : 1)\} \times Y$.

This shows that Y is rationally equivalent to $\varnothing$, for all closed subvariety Y that is not $\mathbb{A}^n$ itself. Apply the previous lemma and observe that the only single closed subvariety of dimension n of $\mathbb{A}^n$ is $\mathbb{A}^n$ itself, so $A(\mathbb{A}^n)$ is just the free abelian group generated by $\mathbb{A}^n$. □

Proposition 3.5. $A_0(\mathbb{P}^n) = \mathbb{Z} \cdot [p]$ *for any closed point $p \in \mathbb{P}^n$ (they are all rationally equivalent).*

Proof. (Adapted from [Gat18, Example 9.1.9.]) First, observe that all closed points are rationally equivalent: let $p, q \in \mathbb{P}^n$ be two closed points, say, $p = (p_0 : p_1 : \cdots : p_n)$ and $q = (q_0 : q_1 : \cdots : q_n)$. Then let $L \subseteq \mathbb{P}^1 \times \mathbb{P}^n$ be a projective line joining p and q, say,

$$L := V_{\mathbb{P}^1 \times \mathbb{P}^n}(\{x_i(sp_j + tq_j) - x_j(sp_i + tq_i) : 0 \leq i, j \leq n\}).$$

Then $L_{(1:0)} = \{(1 : 0)\} \times \{p\}$ and $L_{(0:1)} = \{(0 : 1)\} \times \{q\}$, so p is rationally equivalent to q.

We're left with showing that such class (they're the same) is nonzero. Define an invariant, *degree*, $\deg \colon Z_0(\mathbb{P}^n) \to \mathbb{Z}$, given by $\deg\left(\sum_i n_i \langle Y_i \rangle\right) := \sum_i n_i$. It is an abelian group homomorphism. Observe now that $\deg(B_0(\mathbb{P}^n)) = \{0\}$. (This is not true in general. It is true for $\mathbb{P}^n$ because, considering $B_0(\mathbb{P}^n)$, any closed subvariety of dimension one of $\mathbb{P}^n$ is a projective variety, and the rational functions of a projective variety are fractions for which the numerator and the denominator have the same degree; since the base field is algebraically closed, one can count all the zeros and poles, which corresponds to each irreducible part of the divisor of such rational function) This means, in $\mathbb{P}^n$, the *degree* is preserved by rational equivalence. Since the degree of a closed point is nonzero, we see that the class of any closed point (they're the same) is nonzero. □

Remark. *One can extend this notion and define the* degree *for the closed subvarieties of any dimension. Recall that in our lectures, we've defined the* degree *of a projective scheme of dimension n as $n!$ times the leading coefficient of the Hilbert polynomial of that scheme. Then, define $\deg\left(\sum_i n_i \langle Y_i \rangle\right) := \sum_i n_i \deg(Y_i)$. It turns out that it is still true that such degree is preserved under rational equivalence, when $X = \mathbb{P}^N$ is a projective space. Now we can revisit the example 2.3 and use the above argument to argue that since the* degree *is preserved by rational equivalence in the case of $X = \mathbb{P}^2$, and Y_1 and Y_2 has different degrees, we see that they cannot be rationally equivalent.*

4 Proper Pushforward and Flat Pullback

The following motivating examples are taken from [Gat18, Example 9.1.10. and 9.1.11].

Let X be a scheme. Let $Y \subseteq X$ be a closed subscheme. Let $i \colon Y \hookrightarrow X$ be its inclusion morphism (i.e. closed immersion). Then, we can construct a canonical *pushforward map* $i_* \colon A(Y) \to A(X)$

of Chow groups, defined by sending $[Z]$ to $[Z]$ (if Z is a closed subvariety of Y then it is also a closed subvariety of X). This respects rational equivalence.

Similarly, let X be a scheme. Let $U \subseteq X$ be an open subset. Let $i\colon U \to X$ be its inclusion morphism. Then, we can construct a canonical *pullback map* $i^*\colon A(X) \to A(U)$ by sending $[Z]$ to $[Z \cap U]$ for a closed subvariety Z of X. This respects rational equivalence since $i^* \operatorname{div}(\varphi) = \operatorname{div}(\varphi|_U)$.

We see that in these two cases, we can come up with induced pushforward and induced pullback on Chow groups. Is this the case in general? Unfortunately, it turns out no. Consider the following example.

Example 4.1. *Let $i\colon U \to X$ be an inclusion of an open subset into a scheme. We've seen that i^* is well-defined. However, i_* may not be. Observe the case of $X = \mathbb{P}^1$ and $U = \mathbb{A}^1$. Let $p \in \mathbb{A}^1$ be a point. Then $i_*([p]) = i_*([\varnothing]) = [\varnothing]$ since $A_0(\mathbb{A}^1) = 0$ (recall 3.4). But the class of the same point p in $\mathbb{P}^1$ is nonzero (recall 3.5).*

Similarly, consider the following.

Example 4.2. *Let $i\colon Y \hookrightarrow X$ be an inclusion of a closed subscheme into a scheme. We've seen that i_* is well-defined. However, it is unclear how to define i^*.*

However, we can still go pretty far with some mild assumptions on the maps, but due to limited time and scope, we will only state the conditions and results here.

Recall the notion of *proper morphisms*: a morphism $f\colon X \to Y$ is said to be proper if it is separated, of finite type, and is *universally closed*, i.e., it is closed and for all scheme (not necessarily satisfying our standing assumptions) Z and morphism $g\colon Z \to Y$, the induced base extension $f'\colon X \times_Y Z \to Z$ is also closed (the following commutative diagram might make it clearer).

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y. \end{array}$$

By our standing assumptions from the beginning, all morphisms are automatically separated and of finite type, so a morphism is now *proper* if and only if it is *universally closed*. We have the following:

Theorem 4.3 ([Ful98, Theorem 1.4.]). *If $f\colon X \to Y$ is a proper morphism, then there is an induced $f_*\colon Z_k(X) \to Z_k(Y)$ with $f_*(B_k(X)) \subseteq B_k(Y)$. This induces $f_*\colon A_k(X) \to A_k(Y)$ and hence forms a covariant functor of proper morphisms into Chow groups.*

Proof. See [Ful98, Theorem 1.4.]. □

Warning: the pushforward isn't simply $f_*(V) := \langle f(V)\rangle$. The correct definition involves viewing $K(V)$ as a field extension of $K(f(V))$ (induced by the sheaf pullback $f^\sharp$ and by localizing), then consider $f(V)$: if it has smaller dimension than V, we send it to zero, but if it has the same dimension, we send it to $[K(V)\colon K(f(V))]$ times $\langle f(Z)\rangle$. For more precise statements, see [Gat18, Construction 9.2.9.] or [Ful98, 1.4.].

Now, for the pullback, recall the notion of flatness. An A-module M is flat if $M \otimes_A \bullet$ is an exact functor. A morphism $f\colon X \to Y$ of schemes is *flat* at $p \in X$ if $\mathscr{O}_{X,p}$ is a flat $\mathscr{O}_{Y,f(p)}$-module.[2] A morphism $f\colon X \to Y$ of schemes has *relative dimension* n if for all closed subvariety V of Y, for each irreducible component V' of $f^{-1}(V)$, $\dim V' = \dim V + n$. We have the following pullback result.

Theorem 4.4 ([Ful98, Theorem 1.7.]). *If $f\colon X \to Y$ is a flat morphism of relative dimension n, then, one defines $f^*\colon Z_k(Y) \to Z_{k+n}(X)$ sending $\langle V\rangle$ (where V is a closed subvariety of Y) to $\langle f^{-1}(V)\rangle$, and extend by linearity. Then $f^*(B_k(Y)) \subseteq B_{k+n}(X)$, so it induces $f^*\colon A_k(Y) \to A_{k+n}(X)$, and hence forms a contravariant functor of flat morphisms into Chow groups.*

Proof. See [Ful98, Theorem 1.7.]. □

Although we don't go through this level of generality, we can still arrive at some useful result using only the basic inclusions from the first two examples in this section.

[2]Recall that a morphism $f\colon X \to Y$ comes with a pullback $f^\sharp\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$, and when passed to the stalk, we get $f^\sharp\colon \mathscr{O}_{Y,f(p)} \to \mathscr{O}_{X,p}$. This turns $\mathscr{O}_{X,p}$ into a $\mathscr{O}_{Y,f(p)}$-module.

Theorem 4.5 (Excision)**.** *Let X be a scheme. Let $Y \subseteq X$ be a closed subscheme. Let $U := X \setminus Y$. Let $i\colon Y \to X$ and $j\colon U \to X$ be the inclusion maps. Then the sequence*

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

is exact for all $k \geq 0$. The homomorphism i_* is in general not injective however.

Proof. (1) j^* is surjective: any closed subvariety $Z \subseteq U$ has its closure in X, $\overline{Z}$ as a closed subvariety of X, and we have $j^*\langle\overline{Z}\rangle = \langle\overline{Z} \cap U\rangle = \langle Z\rangle$. This proves the surjectivity.

(2) $\operatorname{im}(i_*) \subseteq \ker(j^*)$: we have, for a closed subvariety Z of Y,

$$j^*(i_*[Z]) = j^*([Z]) = [Z \cap U] = [\varnothing] = 0.$$

(3) $\ker(j^*) \subseteq \operatorname{im}(i_*)$: Let $\alpha \in A_k(X)$ be such that $j^*(\alpha) = 0$ in $A_k(U)$. Say, $\alpha = [\sum_{i=1}^n n_i\langle Z_i\rangle]$ (where Z_i is a closed subvariety of X of dimension k) so that $\sum_{i=1}^n n_i\langle Z_i \cap U\rangle \in B_k(U)$. By definition, this means there exists a collection $(W_j)_{1\leq j\leq m}$ of $(k+1)$-dimensional closed subvarieties of U, and there exists $\varphi_j \in K(W_j)^\times$ for each j, such that

$$\sum_{i=1}^n n_i\langle Z_i \cap U\rangle = \sum_{j=1}^m \operatorname{div}(\varphi_j).$$

Now we try to lift the (W_j, φ_j) into $(\overline{W_j}, \varphi_j)$, where $\overline{W_j}$ is the closure of W_j in X, and $\varphi_j \in K(W_j)^\times = K(\overline{W_j})^\times$. Let

$$\overline{\operatorname{div}}(\varphi_j) := \sum_{\substack{V \subseteq \overline{W_j} \\ \text{closed subvariety of} \\ \text{codimension one}}} \operatorname{ord}_V(\varphi_j)\langle V\rangle \in Z(X)$$

be the lifted div (i.e. its version in the settings of $B_k(X)$). One can compare div and $\overline{\operatorname{div}}$ and observe that any such V corresponds to $V \cap U$ when passed into j^*. This means $\overline{\operatorname{div}}(\varphi_j)$ is basically $\operatorname{div}(\varphi_j)$ (with its subvarieties taken closure in X) plus some extra terms coming from V such that $V \cap U = \varnothing$. That is,

$$\sum_{j=1}^m \overline{\operatorname{div}}(\varphi_j) = \sum_{i=1}^n n_i\langle Z_i\rangle + \sum_{\substack{\text{some other } V \\ \text{with } V\cap U=\varnothing}} n_V\langle V\rangle.$$

For these $V \cap U = \varnothing$, we see that $V \subseteq X \setminus U = Y$, so they are in $\operatorname{im}(i_*)$. Passing everything by quotient into $A_k(X)$, we see that

$$\underbrace{\left[\sum_{j=1}^m \overline{\operatorname{div}}(\varphi_j)\right]}_{0} = \underbrace{\left[\sum_{i=1}^n n_i\langle Z_i\rangle\right]}_{\alpha} + i_*(\beta),$$

for some $\beta \in A_k(Y)$. This shows that $\alpha \in \operatorname{im}(i_*)$.

This completes the proof. □

Remark. *Apart from excision, there are more ways to compute Chow groups. Most notably the Mayer–Vietoris exact sequence and affine stratifications. We refer to [EH16] for reference on this. However, it is still pretty hard to compute Chow groups in general.*

Proposition 4.6. *Extending the result of 3.5, $A_k(\mathbb{P}^n) \cong \mathbb{Z}$ for each $0 \leq k \leq n$. One can take $[Z] \to \deg(Z)$ to be such isomorphism.*

Proof. Recall that $A_k(\mathbb{P}^n) = 0$ for all $k > n$. We do induction on n. By 4.5, we have an exact sequence

$$A_k(\mathbb{P}^{n-1}) \to A_k(\mathbb{P}^n) \to A_k(\mathbb{A}^n) \to 0.$$

Recall 3.4 to see that $A_k(\mathbb{A}^n) = 0$ for all $0 \leq k < n$. This means $A_k(\mathbb{P}^{n-1}) \to A_k(\mathbb{P}^n)$ is surjective. By induction hypothesis, $A_k(\mathbb{P}^{n-1}) \cong \mathbb{Z}$, so $\mathbb{Z} \to A_k(\mathbb{P}^n) \to 0$ exact. Since deg is preserved under $i_*\colon A_k(\mathbb{P}^{n-1}) \to A_k(\mathbb{P}^n)$, we see that $A_k(\mathbb{P}^n)$ must be torsion-free. It is nonzero since one can find $\mathbb{P}^k \subseteq \mathbb{P}^n$ with degree one. So $A_k(\mathbb{P}^n) \cong \mathbb{Z}$. Since $\deg(B_k(\mathbb{P}^n)) = \{0\}$, we see that deg induces an isomorphism from $A_k(\mathbb{P}^n) \to \mathbb{Z}$. The case of $k = n$ gives

$$0 \to A_n(\mathbb{P}^n) \to \mathbb{Z} \to 0,$$

so $A_n(\mathbb{P}^n) \cong \mathbb{Z}$ immediately, and the only closed n-dimensional subvariety of $\mathbb{P}^n$ is itself. □

A Appendix: Divisors

We do a brief review of the theory of Weil divisors and Cartier divisors. See [Har77, II.6] and [Vak25, 15.4.11.1] for more detailed discussions.

Recall that for a scheme, we can define the *Weil divisors*, $\operatorname{Weil}(X)$, as the free abelian group generated by closed subvarieties of codimension one. (So if X has pure dimension n, then $\operatorname{Weil}(X) = Z_{n-1}(X)$.) By similar process as in defining div before, if X is a variety, we define $\operatorname{div}\colon K(X)^\times \to \operatorname{Weil}(X)$ as

$$\operatorname{div}(\varphi) := \sum_{\substack{Y \subseteq X \\ \text{closed subvariety of} \\ \text{codimension one}}} \operatorname{ord}_Y(\varphi)\langle Y \rangle \in \operatorname{Weil}(X),$$

and let $\operatorname{WeilPrin}(X)$ be the free abelian group generated by the image of div. Put $\operatorname{WeilCl}(X) := \operatorname{Weil}(X)/\operatorname{WeilPrin}(X)$.

For a general scheme X, we define the *sheaf of total quotient ring*, $\mathscr{K}_X$, as the sheafification of the presheaf $\mathscr{K}_X^{\mathrm{pre}}$, defined by the following procedure. For an open set $U \subseteq X$, let $S(U) := \{f \in \mathscr{O}_X(U) \colon f_x \in \mathscr{O}_{X,x} \text{ is not a zerodivisor for all } x \in U\}$ and put $\mathscr{K}_X^{\mathrm{pre}}(U) := (S(U))^{-1}\mathscr{O}_X(U)$. Then, we define the *Cartier divisors*, $\operatorname{Cartier}(X)$, as the global sections $\Gamma(X, \mathscr{K}_X^\times/\mathscr{O}_X^\times)$. We define $\operatorname{CartierPrin}(X)$ as the image of the natural map $\Gamma(X, \mathscr{K}_X^\times) \to \Gamma(X, \mathscr{K}_X^\times/\mathscr{O}_X^\times)$, and define $\operatorname{CartierCl}(X) := \operatorname{Cartier}(X)/\operatorname{CartierPrin}(X)$.

Remark. *Practically, we describe an element of* $\operatorname{Cartier}(X)$ *by* $\{(U_i, f_i)\}_{i\in I}$ *where* $(U_i)_{i\in I}$ *is an open cover of* X*, and* $f_i \in \mathscr{K}_X(U_i)^\times$ *such that* $\frac{f_i}{f_j} \in \mathscr{O}_X^\times(U_i \cap U_j)$ *for all* $i, j \in I$*. One can see that by the sheaf property, such description agrees (modulo* $\mathscr{O}_X^\times$*) on the overlaps, hence glues to a global section of* $\mathscr{K}_X^\times/\mathscr{O}_X^\times$*. The converse holds by restricting a representative of a global section to any open cover.*

A line bundle on a scheme X is a sheaf of $\mathscr{O}_X$-module which is locally free of rank one. If $\mathscr{L}$ is a line bundle, then $\mathscr{L}^\vee := \mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$ is another $\mathscr{O}_X$-module with $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}^\vee \cong \mathscr{O}_X$. Hence the collection of line bundles on X modulo isomorphism is an abelian group. We call it the *Picard group* and denote it by $\operatorname{Pic}(X)$.

We recall the following summary (the bottom right $\operatorname{WeilCl}(X)$ is well-defined only if X is integral).

$$\begin{array}{ccccccc}
 & & \operatorname{LineBundleWithRatSection}(X)/\cong & \xleftarrow{(\mathscr{O}_X(D),s) \leftarrow\!\shortmid D} & \operatorname{Cartier}(X) & \longrightarrow & \operatorname{Weil}(X) \\
 & & \Big\downarrow{\scriptstyle \text{forget section}} & & \Big\downarrow{\scriptstyle \text{quot}} & & \Big\downarrow{\scriptstyle \text{quot}} \\
\operatorname{Pic}(X) & = & \operatorname{LineBundle}(X)/\cong & \xleftarrow[\mathscr{O}_X(D) \leftarrow\!\shortmid D]{} & \operatorname{CartierCl}(X) & \longrightarrow & \operatorname{WeilCl}(X).
\end{array}$$

The map $\operatorname{Cartier}(X) \to \operatorname{Weil}(X)$ is defined as the following: for a Cartier divisor $\{(U_i, f_i)\}_{i\in I}$, for each closed subvariety Y of X of codimension one, take the coefficient n_Y to be $\operatorname{ord}_Y(f_i)$ for any $i \in I$ such that $U_i \cap Y \neq \varnothing$. If $j \in I$ is another such index, observe that $\operatorname{ord}_Y(f_i/f_j) = 0$ because f_i/f_j is invertible on $U_i \cap U_j$. This shows that $\sum_{Y \subseteq X \text{ closed subvariety of codimension one}} n_Y\langle Y\rangle$ is well-defined as a Weil divisor corresponding to the Cartier divisor $\{(U_i, f_i)\}_{i\in I}$.

If X is an integral scheme, then the middle maps are isomorphisms [Har77, Proposition II.6.15.]. If X is integral and all local rings of X are UFDs (i.e. locally factorial), then the maps in the right column are isomorphisms [Har77, Proposition II.6.11.]. Since a smooth scheme is regular [Vak25, 13.2.7.(b)], and a regular scheme is factorial (Auslander–Buchsbaum Theorem), we see that being smooth is enough to conclude that the maps in the right column are isomorphisms.

Observe further that if X is an integral scheme of pure dimension n, then $\operatorname{Weil}(X) = Z_{n-1}(X)$ and $\operatorname{WeilCl}(X) = A_{n-1}(X)$.

B Appendix: Intersection with Divisors

We're now ready to define the intersection product following [Gat18, Definition 9.4.1.].

Definition. *Let* X *be a variety. Let* $V \subseteq X$ *be a* k*-dimensional closed subvariety with inclusion morphism* $i\colon V \to X$*. Let* D *be a Cartier divisor on* X*. We define the intersection product* $D \cdot \langle V \rangle \in A_{k-1}(X)$ *to be*

$$D \cdot \langle V \rangle := i_*[i^*\mathscr{O}_X(D)].$$

That is, send D through the following, where the i^* is the pullback of Picard groups and the second i_* is the proper pushforward of Chow rings:

$$\begin{array}{ccccccc}
\mathrm{Cartier}(X) & \longrightarrow & \mathrm{Pic}(X) & & & & \\
 & & \big\downarrow{\scriptstyle i^*} & & & & \\
 & & \mathrm{Pic}(V) & \xrightarrow{\sim} & \mathrm{CartierCl}(V) & \longrightarrow & \mathrm{WeilCl}(V) & = & A_{k-1}(V) \\
 & & & & & & & & \big\downarrow{\scriptstyle i_*} \\
 & & & & & & & & A_{k-1}(X).
\end{array}$$

Extending linearly on the second argument, and observing that it doesn't depend on the class of D within $\mathrm{CartierCl}(X)$*, we obtain a map* $\cdot\colon \mathrm{Pic}(X) \times Z_k(X) \to A_{k-1}(X)$.

We mention the following two results, whose proofs can be found in [Ful98, Proposition 9.4.4. and Corollary 9.4.5.].

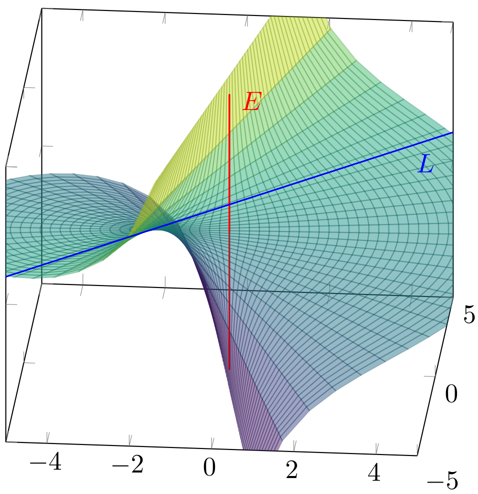
Theorem B.1 (Commutativity of the intersection product)**.** *Let X be a smooth n-dimensional variety. Let $D_1, D_2 \in \mathrm{Cartier}(X)$, with associated Weil divisors $\langle D_1\rangle, \langle D_2\rangle \in \mathrm{Weil}(X)$. Then*

$$D_1 \cdot \langle D_2\rangle = D_2 \cdot \langle D_1\rangle \in A_{n-2}(X).$$

Corollary B.2. *Let X be a smooth n-dimensional variety. The intersection product* $\mathrm{Pic}(X) \times Z_k(X) \to A_{k-1}(X)$ *descends to* $\mathrm{Pic}(X) \times A_k(X) \to A_{k-1}(X)$.

C Appendix: Example on $\widetilde{\mathbb{P}^2}$

(Example taken from [Gat18, Example 9.2.14.])



Let $\widetilde{\mathbb{P}^2}$ be the blowup of $\mathbb{P}^2$ at the point $p = (1:0:0)$. Let us compute its Chow group. By excision, we get an exact sequence

$$A_k(\{p\}) \to A_k(\mathbb{P}^2) \to A_k(\mathbb{P}^2 \setminus \{p\}) \to 0.$$

We immediately get $A_1(\mathbb{P}^2 \setminus \{p\}) \cong A_2(\mathbb{P}^2 \setminus \{p\}) \cong \mathbb{Z}$. For A_0 we have

$$\mathbb{Z} \to \mathbb{Z} \to A_0(\mathbb{P}^2 \setminus \{p\}) \to 0,$$

but we knew that the first map (pushforward of inclusion) in this case gives an isomorphism, so by exactness, we get $A_0(\mathbb{P}^2 \setminus \{p\}) = 0$. So we get $A(\mathbb{P}^2 \setminus \{p\}) = \mathbb{Z} \cdot [\xi] \oplus \mathbb{Z} \cdot [\zeta]$ where ξ is any one dimensional closed subvariety and ζ is any two dimensional closed subvariety. Now we look at the blowup $\widetilde{\mathbb{P}^2}$ and get

$$\begin{array}{ccccccc}
A_k(E) & \longrightarrow & A_k(\widetilde{\mathbb{P}^2}) & \longrightarrow & A_k(\widetilde{\mathbb{P}^2} \setminus E) & \longrightarrow & 0 \\
\| & & & & \| & & \\
A_k(\mathbb{P}^1) & & & & A_k(\mathbb{P}^2 \setminus \{p\}). & &
\end{array}$$

This shows that $A_0(\widetilde{\mathbb{P}^2}) \cong A_2(\widetilde{\mathbb{P}^2}) \cong \mathbb{Z}$. For A_1, observe that a closed subvariety V of $\widetilde{\mathbb{P}^2}$ either intersects with $U := \widetilde{\mathbb{P}^2} \setminus E$ (which means $V \cap U$ restricts to a closed subvariety of U, a class of $A_k(U)$), or doesn't intersect with U (which means it is a closed subvariety of E, its class coming from $A_k(E)$). This means $A_1(\widetilde{\mathbb{P}^2})$ is generated by an element of $A_1(E)$ and one coming from $A_1(U)$, possibly modulo some relation.

Let us show that there is in fact no relation. Suppose there is a relation, say, $n[L]+m[E] = 0$ in $A_1(\widetilde{\mathbb{P}^2})$. Then let $\pi\colon \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the projection to the base. This is a proper map, so $\pi_*[L] = [H]$ and $\pi_*[E] = 0$, where $[H]$ is the class of a line in $\mathbb{P}^2$. We see that

$$0 = \pi_*(0) = \pi_*(n[L] + m[E]) = n[H] \in A_1(\mathbb{P}^2),$$

so $n = 0$.

Now let $\varphi\colon \widetilde{\mathbb{P}^2} \to E$ be the morphism that is identity on E and "project" every point outside E to E. Again this is a proper morphism, and $p_*[L] = 0$ and $p_*[E] = [E]$. Therefore,

$$0 = \varphi_*(0) = \varphi_*(n[L] + m[E]) = m[E] \in A_1(E) = A_1(\mathbb{P}^1).$$

This shows that $m = 0$.

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